

# Self-similar cosmologies in $5D$ : Our universe as a topological separation from an empty $5D$ Minkowski space.

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## Abstract

In this paper we find the most general self-similar, homogeneous and isotropic, Ricci flat cosmologies in  $5D$ . These cosmologies show a number of interesting features: (i) the field equations allow a complete integration in terms of one arbitrary function of the similarity variable, and a free parameter; (ii) the three-dimensional spatial surfaces are flat; (iii) the extra dimension is spacelike; (iv) the general solution is Riemann-flat in  $5D$  but curved in  $4D$ , which means that an observer confined to  $4D$  spacetime can relate this curvature to the presence of matter, as determined by the Einstein equations in  $4D$ . We show that these cosmologies can be interpreted, or used, as  $5D$  Riemann-flat embeddings for spatially-flat FRW cosmologies in  $4D$ . In this interpretation our universe arises as a topological separation from an empty  $5D$  Minkowski space, as envisioned by Zeldovich.

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# 1 Introduction

The question of whether there was a Beginning of the Universe, is a truly challenging puzzle for physics and for philosophy [1]. In classical four-dimensional general relativity, the singularity theorems imply that the big bang was a unique birth event for a universe filled with matter satisfying the weak, dominant and strong energy conditions [2]-[3]. The big bang model postulates that our observable universe originated in a singularity sometime between 10 and 20 billion years ago. Before the bang nothing existed, not space, time, matter, or energy.

Astronomical observations support the notion that our universe began at some point in the finite past [4]-[6]. The question is *how* did it begin. Twenty seven years ago, in a paper concerning the idea of a spontaneous birth of our universe, Ya. B. Zeldovich observed that “there is a certain arbitrariness and fuzziness in the very concept of spontaneous birth” [7]. He inquired whether spontaneous birth emerges (i) “out of nothing”, or (ii) in a space of more dimensions, or (iii) as a topological separation from an initially given empty Minkowski space.

Today, in the literature we find a huge number of papers examining theories for the birth of our universe along the lines (i) and (ii) envisioned by Zeldovich<sup>1</sup>. In fact, there are so many interesting papers and books about these topics that for us here is impossible, and far beyond the scope of this work, to give a thorough list of references. Therefore we restrict ourselves to mention just few representative works where one can find a detailed bibliography: tunneling from nothing [11]-[17]; spontaneous creation of the braneworld [18]-[21]; induced matter in Kaluza-Klein gravity and STM (Space-Time-Matter) theories [22]-[26].

Concerning the third alternative mentioned by Zeldovich, the hypothesis that singularities in  $4D$  are induced by the separation of spacetime from the other dimensions has been examined by the present author [27]; Seahra and Wesson have thoroughly investigated the structure of the big bang from a five-dimensional embedding [22] for the standard spatially-flat  $4D$  FRW models [28]. Besides these studies, there are only few more works related to this alternative [29]-[33]. The aim of this paper is to present a general class of spatially homogeneous and isotropic solutions of the Einstein field equations in  $5D$  that is relevant to this alternative.

To be more precise, in many Kaluza-Klein and braneworld theories the higher dimensional space is assumed to be either Ricci-flat or anti-de Sitter. In this work we develop a family of metrics that are Ricci-flat and Riemann-flat in  $5D$ , but whose four-dimensional subspaces are curved. Therefore, they are all equivalent to an empty Minkowski space in  $5D$ . However, since the Riemann tensor of four-dimensional subspaces is non-vanishing, for an observer confined to  $4D$  the spacetime is not empty but contains matter as determined by the Einstein equations in  $4D$ .

The solutions arise from the observation that cosmological models are self-similar [34]. In order to illustrate this in  $5D$ , let us consider the metric<sup>2</sup> [22]

$$dS^2 = Cy^2 dt^2 - Dt^{2/\alpha} y^{2/(1-\alpha)} [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)] - \frac{C\alpha^2 t^2}{(1-\alpha)^2} dy^2, \quad (1)$$

where  $\alpha$  is a free parameter and  $C$  as well as  $D$  are constants with the appropriate units. The properties of this metric have widely been discussed in the literature, e.g., [24], [28], [35]. It is Riemann-flat and apparently empty in  $5D$ , but reduces to the well-known  $4D$  FRW models with flat  $3D$  sections on  $y = \text{constant}$  hypersurfaces (henceforth denoted by  $\Sigma_y$ ). It is easy to verify that (1) admits a homothetic Killing vector, viz.,

$$\mathcal{L}_\xi g_{AB} = 2g_{AB}, \quad \text{with} \quad \xi^A = \left( \frac{\alpha t}{(2\alpha-1)}, r, 0, 0, \frac{(\alpha-1)y}{(2\alpha-1)} \right). \quad (2)$$

With the transformation of coordinates

$$t \rightarrow \bar{t}^{\alpha/(2\alpha-1)}, \quad r \rightarrow \bar{r}, \quad y \rightarrow \bar{y}^{(\alpha-1)/(2\alpha-1)}, \quad (3)$$

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<sup>1</sup>It should be mentioned that in addition to the theories envisioned by Zeldovich, there are many other ones, e.g., evolution of our universe from a  $4D$  Minkowski spacetime [8], [9]. More recently, we find black hole generation of universes; universes spontaneously creating other universes; creation of universes by intelligent life. For a wonderful popular review, see for example [10]

<sup>2</sup>Conventions: Throughout the paper we use geometric units where  $c = G = 1$ ;  $t = x^0$ ,  $r = x^1$ ,  $\theta = x^2$  and  $\phi = x^3$  are the usual coordinates for a spacetime with spherically symmetric spatial sections;  $y = x^4$  represents the coordinate along the extra dimension; the signature of the  $5D$  metric is  $(+, -, -, -, \epsilon)$  where  $\epsilon$  can be either  $-1$  or  $+1$  depending on whether the extra dimension is spacelike or timelike. The range of tensor indices is  $A, B, \dots = 0 - 4$  and  $\mu, \nu, \dots = 0 - 3$ .

we have

$$\xi^A \rightarrow \bar{\xi}^A = (\bar{t}, \bar{r}, 0, 0, \bar{y}) \quad (4)$$

and the metric (1) becomes

$$dS^2 = \bar{A} \xi^{2(1-\alpha)/(2\alpha-1)} d\bar{t}^2 - \bar{B} \xi^{2/(2\alpha-1)} [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)] - \bar{A} \xi^{2\alpha/(2\alpha-1)} d\bar{y}^2, \quad (5)$$

where  $\bar{A}$  and  $\bar{B}$  are dimensionless constants and

$$\xi = \frac{\bar{t}}{\bar{y}}. \quad (6)$$

The above clearly illustrates the self-similar nature of (1). In this work we extend our previous studies [22] and [34] to obtain a general class of self-similar cosmologies that are Ricci-flat and Riemann-flat in  $5D$ , but curved in  $4D$ . Consequently, they can be used, or interpreted, as  $5D$  Riemann-flat embeddings for our  $4D$  universe, which is consistent with Zeldovich's notion that our universe could have emerged as a topological separation from an initially given empty Minkowski space in more than four dimensions.

The paper is organized as follows. In Section 2, we deduce the equations with self-similarity in  $5D$ . In Section 3, we obtain the general solutions to these equations; they contain an arbitrary function of the similarity variable, and a free parameter. In Section 4, we focus our attention on a class of solutions that admits a particularly simple homothetic Killing vector in  $5D$ , and study their possible application in  $4D$ . We use the flexibility of the solution to develop some cosmological models in  $4D$ . In Section 5, we present a summary of our results and propose several extensions for this work. Finally, in the Appendix, we discuss the homothetic symmetry on  $\Sigma_y$ .

## 2 Field equations

In this section we present the field equations for self-similar cosmologies in  $5D$ . We start with the line element for a spacetime that has spatial spherical symmetry

$$dS^2 = e^{\nu(r,t,y)} dt^2 - e^{\lambda(r,t,y)} dr^2 - R^2(r,t,y) [d\theta^2 + \sin^2 \theta d\phi^2] + \epsilon \Phi^2(r,t,y) dy^2. \quad (7)$$

For cosmological models we assume spatial homogeneity, which means that the metric is invariant under spatial translations. However, we do not make any assumption regarding the curvature of  $3D$ -space. Now, as illustrated above (5), by a suitable transformation of coordinates in a self-similar model all the dimensionless quantities can be put in a form where they are functions only of a single variable (say  $\xi$ ) [36]-[43]. Thus, in the case under consideration, in “self-similar” coordinates  $\bar{t}, \bar{r}$ , and  $\bar{y}$ , the line element (7) can be written as

$$dS^2 = e^{\nu(\xi)} d\bar{t}^2 - e^{\lambda(\xi)} d\bar{r}^2 - \bar{r}^2 e^{\mu(\xi)} [d\theta^2 + \sin^2 \theta d\phi^2] + \epsilon \Phi^2(\xi) d\bar{y}^2. \quad (8)$$

On the other hand, we have *no* reason to set  $\xi = \bar{t}/\bar{y}$  as in (6). In principle  $\xi$  can be any function of  $\bar{t}$  and  $\bar{y}$ , namely

$$\xi = \xi(\bar{t}, \bar{y}). \quad (9)$$

With this choice the line element (8) is self-similar but not necessarily admits a homothetic Killing vector (More comments about this at the end of Section 3). In what follows we are going to suppress the bar over the self-similar coordinates.

The metric functions in (7) and (8) have to satisfy the  $5D$  Einstein field equations in apparent vacuum, which in terms of the Ricci tensor are

$$R_{AB} = 0. \quad (10)$$

At once we note that  $R_{01} = 0$  requires<sup>3</sup>

$$\lambda_\xi = \mu_\xi \quad (11)$$

from which we get

$$e^{\mu(\xi)} = C_0 e^{\lambda(\xi)}, \quad (12)$$

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<sup>3</sup>In what follows  $f_\xi$  denotes derivative of  $f$  with respect to  $\xi$ ; dots and primes stand for derivatives with respect to  $t$  and  $y$ , respectively.

where  $C_0$  is a constant of integration. Now, from  $R_1^1 = R_2^2 = R_3^3$  it follows that

$$C_0 = 1. \quad (13)$$

Consequently, the requested self-similarity (8), together with the field equations (10), demand  $R = re^{\lambda/2}$ . Therefore, the line element for self-similar cosmological models in  $5D$  reduces to

$$dS^2 = e^{\nu(\xi)} dt^2 - e^{\lambda(\xi)} [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)] + \epsilon \Phi^2(\xi) dy^2. \quad (14)$$

This implies that in self-similar cosmologies the  $3D$  spatial sections ( $t = \text{constant}$ ,  $y = \text{constant}$ ) are flat, in agreement with astrophysical data [4]-[6]. For this metric the Ricci tensor in  $5D$  has four non-trivial, independent, components. These are

$$R_{00}, \quad R_{04}, \quad R_{11} = \frac{R_{22}}{r^2} = \frac{R_{33}}{r^2 \sin^2 \theta}, \quad R_{44}. \quad (15)$$

A simple analysis of equation  $R_{04} = 0$ , indicates that self-similarity requires that the ratio  $[\dot{\xi}'/(\xi'\dot{\xi})]$  be some function of  $\xi$ . Clearly, *any* separable function of  $t$  and  $y$  will do the job. Therefore, without loss of generality we can set

$$\xi = \frac{T(t)}{Y(y)}, \quad (16)$$

where  $T$  and  $Y$  are some functions to be determined by the field equations in  $5D$ . Consequently, we have four differential equations for five unknown, viz.,

$$\nu(\xi), \quad \lambda(\xi), \quad \Phi(\xi), \quad T(t), \quad Y(y). \quad (17)$$

We note that  $\epsilon$  is also to be defined from the field equations. We will see that we can solve the field equations (10) in terms of one arbitrary function and a free parameter. The four equations to be integrated are:

$$1. \quad R_{00} = 0,$$

$$\begin{aligned} & \epsilon \Phi^2 \dot{T}^2 \left( 6\lambda_{\xi\xi} + 3\lambda_{\xi}^2 - 3\nu_{\xi}\lambda_{\xi} + \frac{4\Phi_{\xi\xi}}{\Phi} - \frac{2\nu_{\xi}\Phi_{\xi}}{\Phi} \right) + 2\epsilon \Phi^2 \ddot{T}Y \left( 3\lambda_{\xi} + \frac{2\Phi_{\xi}}{\Phi} \right) + \\ & Y'^2 e^{\nu} \xi^2 \left( 2\nu_{\xi\xi} + \nu_{\xi}^2 + 3\nu_{\xi}\lambda_{\xi} - \frac{2\nu_{\xi}\Phi_{\xi}}{\Phi} \right) + 2Y'^2 e^{\nu} \left( 2 - \frac{YY''}{Y'^2} \right) \xi \nu_{\xi} = 0, \end{aligned} \quad (18)$$

$$2. \quad R_{04} = 0,$$

$$2\lambda_{\xi\xi} + \lambda_{\xi}^2 + \frac{2\lambda_{\xi}}{\xi} - \nu_{\xi}\lambda_{\xi} - \frac{2\lambda_{\xi}\Phi_{\xi}}{\Phi} = 0. \quad (19)$$

$$3. \quad R_{11} = R_{22}/r^2 = R_{33}/(r^2 \sin^2 \theta) = 0,$$

$$\begin{aligned} & \epsilon \Phi^2 \dot{T}^2 \left( 2\lambda_{\xi\xi} + 3\lambda_{\xi}^2 - \nu_{\xi}\lambda_{\xi} + \frac{2\lambda_{\xi}\Phi_{\xi}}{\Phi} \right) + 2\epsilon \Phi^2 \ddot{T}Y \lambda_{\xi} + \\ & Y'^2 e^{\nu} \xi^2 \left( 2\lambda_{\xi\xi} + 3\lambda_{\xi}^2 + \nu_{\xi}\lambda_{\xi} - \frac{2\lambda_{\xi}\Phi_{\xi}}{\Phi} \right) + 2Y'^2 e^{\nu} \left( 2 - \frac{YY''}{Y'^2} \right) \xi \lambda_{\xi} = 0. \end{aligned} \quad (20)$$

$$4. \quad \text{Finally, } R_{44} = 0,$$

$$\begin{aligned} & 2\epsilon \Phi^2 \dot{T}^2 \left( \frac{2\Phi_{\xi\xi}}{\Phi} - \frac{\nu_{\xi}\Phi_{\xi}}{\Phi} + \frac{3\lambda_{\xi}\Phi_{\xi}}{\Phi} \right) + 4\epsilon \Phi^2 \ddot{T}Y \left( \frac{\Phi_{\xi}}{\Phi} \right) + \\ & e^{\nu} Y'^2 \xi^2 \left( 2\nu_{\xi\xi} + \nu_{\xi}^2 - \frac{2\nu_{\xi}\Phi_{\xi}}{\Phi} + 6\lambda_{\xi\xi} + 3\lambda_{\xi}^2 - \frac{6\lambda_{\xi}\Phi_{\xi}}{\Phi} \right) + 2Y'^2 e^{\nu} \left( 2 - \frac{YY''}{Y'^2} \right) \xi (\nu_{\xi} + 3\lambda_{\xi}) = 0. \end{aligned} \quad (21)$$

### 3 Integrating the field equations

Let us first notice that (19) can be easily integrated as

$$\xi^2 \lambda_\xi^2 = C^2 \Phi^2 e^{(\nu-\lambda)}, \quad (22)$$

where  $C$  is a constant of integration. On the other hand, the field equations (18), (20) and (21) have the following structure

$$\left( \frac{\dot{T}^2}{Y'^2} \right) F(\xi) + \left( \frac{\ddot{T}Y}{Y'^2} \right) G(\xi) + H(\xi) + I(\xi) \left( 2 - \frac{YY''}{Y'^2} \right) = 0, \quad (23)$$

where  $F$ ,  $G$ ,  $H$  and  $I$  symbolize the corresponding functions of  $\xi$  in these equations. Therefore, in order to preserve the self-similar symmetry, we have to require

$$2 - \frac{YY''}{Y'^2} = l, \quad (24)$$

where  $l$  is a separation constant. Integrating this expression we obtain

$$Y' = \alpha Y^{(2-l)}, \quad (25)$$

where  $\alpha$  is a constant of integration. Thus, we find

$$\left( \frac{\dot{T}^2}{Y'^2} \right) = \left[ \frac{\dot{T}^2}{\alpha^2 T^{(4-2l)}} \right] \xi^{(4-2l)}, \quad \text{and} \quad \left( \frac{\ddot{T}Y}{Y'^2} \right) = \left[ \frac{\ddot{T}}{\alpha^2 T^{(3-2l)}} \right] \xi^{(3-2l)}. \quad (26)$$

Consistency of (23) demands the quantities inside the square brackets to be constants, which requires

$$T \sim \begin{cases} t^{1/(l-1)} & \text{for } l \neq 1, \\ e^{\beta t} & \text{for } l = 1, \end{cases} \quad (27)$$

where  $\beta$  is some constant. Consequently, without loss of generality we can set

$$\xi = \begin{cases} \left( \frac{t}{y} \right)^{1/(l-1)} & \text{for } l \neq 1, \\ \left( \frac{e^{\beta t}}{e^{\alpha y}} \right) & \text{for } l = 1. \end{cases} \quad (28)$$

Let us now calculate  $(R_{00} - R_{44})$ , and use (19) to express the second derivative  $\lambda_{\xi\xi}$  in terms of the first derivatives of  $\nu$  and  $\Phi$ . As a result we obtain

$$\epsilon \Phi^2 (T\ddot{T} - \dot{T}^2) = (Y'^2 - YY'') \xi^2 e^\nu. \quad (29)$$

**Solution for  $l \neq 1$ :** In this case (29) yields

$$\Phi^2 = (-\epsilon) \xi^{2(l-1)} e^\nu. \quad (30)$$

From this expression, it follows that the extra dimension should be spacelike, i.e.,  $\epsilon = -1$ . Now, from (22) and (30) we get

$$e^\nu = \left( \frac{1}{C} \right) \xi^{(2-l)} \lambda_\xi e^{\lambda/2}. \quad (31)$$

In summary, we have found that the line element (14) with

$$e^\nu = \left( \frac{1}{C} \right) \xi^{(2-l)} \lambda_\xi e^{\lambda(\xi)/2}, \quad \Phi^2 = \xi^{2(l-1)} e^{\nu(\xi)}, \quad \xi = \left( \frac{t}{y} \right)^{1/(l-1)}, \quad \text{and} \quad \epsilon = -1, \quad (32)$$

is a solution of the 5D Ricci-flat equations for any value of the parameter  $l \neq 1$ , constant  $C$ , and arbitrary function  $\lambda = \lambda(\xi)$ . In addition, it admits a homothetic Killing vector in 5D, viz.,

$$\mathcal{L}_\xi g_{AB} = 2g_{AB}, \quad \text{with} \quad \xi^A = (t, r, 0, 0, y). \quad (33)$$

It should be noted that (32) includes the family of 5D metrics given by (1), or (5) for the particular choice  $e^{\lambda(\xi)} = B\xi^{2/(2\alpha-1)}$  and  $l = 2$ .

**Solution for  $l = 1$ :** In this case the field equations require  $\beta = \alpha$ . Therefore, the solution is

$$e^\nu = \left(\frac{1}{C}\right) \xi \lambda_\xi e^{\lambda(\xi)/2}, \quad \Phi^2 = e^{\nu(\xi)}, \quad \xi = \left(\frac{e^{\alpha t}}{e^{\alpha y}}\right), \quad \text{and} \quad \epsilon = -1. \quad (34)$$

It is obvious that this solution does not admit a *simple* 5D homothetic vector as (2) or (33) (See below).

### 3.1 The Riemann tensor in 5D

It can be verified that, for both solutions, all the components of the 5D Riemann tensor  $R_{ABCD}$  vanish identically. Therefore, they are equivalent to an empty Minkowski space in 5D ( $\mathcal{M}_5$ ). Consequently, there exist some coordinate transformation

$$\tau = \tau(t, r, y), \quad R = R(t, r, y), \quad \psi = \psi(t, r, y), \quad (35)$$

that brings (32) and (34) to the line element in  $\mathcal{M}_5$

$$dS^2 = \eta_{AB} dx^A dx^B = d\tau^2 - dR^2 - R^2 (d\theta^2 + \sin^2 \theta d\phi^2) - d\psi^2, \quad (36)$$

in Minkowski coordinates  $x^A = (\tau, R, \theta, \phi, \psi)$ . For the particular solution (4), the explicit coordinate transformation from (1) to (36) is known and has been amply discussed in the literature, see for example [28] and references therein.

Before going on, and in order to avoid misunderstandings, we should comment about the homothetic nature of the above solutions. Certainly, any 5D vector of the form

$$\xi_{\mathcal{M}}^A = \{[\tau + f_1(R, \psi)], R, 0, 0, [\psi + f_2(\tau, R)]\}, \quad (37)$$

where  $f_1$  and  $f_2$  are arbitrary functions of their arguments, is a homothetic Killing vector for (36), viz.,  $\mathcal{L}_{\xi_{\mathcal{M}}} \eta_{AB} = 2\eta_{AB}$ . Thus, the Riemann-flat solutions (32) and (34) admit an infinite number of homothetic killing vectors, which correspond to the nondenumerable infinity of choices of  $f_1$  and  $f_2$  in (37).

In this context, the difference between (32) and (34) is that, among all the possible choices, in self-similar coordinates  $t, r$  and  $y$  there exists a very simple homothetic killing vector for the first solution, namely  $\xi^A = (t, r, 0, 0, y)$ , while for the second solution the homothetic vectors (in self-similar coordinates) are much more complicated.

## 4 Interpretation in 4D

In this section we discuss possible four-dimensional interpretations of the 5D line element (32). For simplicity we set

$$l = 2, \quad (38)$$

in such a way that the similarity variable takes the form  $\xi = t/y$ , as in (6). Thus, in what follows we consider the 5D metric

$$dS^2 = \left(\frac{1}{C}\right) \lambda_\xi e^{\lambda(\xi)/2} dt^2 - e^{\lambda(\xi)} [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)] - \left(\frac{1}{C}\right) \xi^2 \lambda_\xi e^{\lambda(\xi)/2} dy^2, \quad (39)$$

with

$$\xi = \frac{t}{y}. \quad (40)$$

The first important question is how to recover our  $4D$  spacetime from  $5D$ . The most popular approach is to assume that our  $4D$  spacetime is a hypersurface  $\Sigma_y : y = y_0 = \text{constant}$ , which is orthogonal to the  $5D$  unit-vector

$$n^A = \frac{\delta_4^A}{\Phi} \quad (41)$$

along the extra dimension, although a dynamical foliation is also possible [44], [45]. The second important question is how to construct the metric of the physical spacetime from the one induced on  $\Sigma_y$ . There are various approaches in the literature: (i) the canonical metric, which assumes  $\Phi = 1$  and factorizes the  $4D$  part of the  $5D$  metric by an  $y^2$  term [46]-[49]; (ii) the conformal approach, where the  $4D$  part of the  $5D$  metric is factorized by an  $\Phi^N$  term (see for example [50] and references therein); and (iii) the one where the spacetime metric is identified with the metric induced on  $\Sigma_y$ .

An exhaustive treatment of all possibilities is beyond the scope of this work. We present here an introductory analysis where we follow the approach (iii) mentioned above. The induced metric  $h_{\alpha\beta}$  on hypersurfaces  $\Sigma_y$  is just the  $4D$  part of the  $5D$  metric (39), viz.,

$$ds^2 = h_{\mu\nu} dx^\mu dx^\nu = \left( \frac{1}{C} \right) \lambda_\xi e^{\lambda(\xi)/2} dt^2 - e^{\lambda(\xi)} [dr^2 + r^2 (d\theta^2 + \sin^2 \phi)]. \quad (42)$$

We immediately notice two things:

Firstly, that the spacetime metric  $h_{\alpha\beta}$  is *not* self-similar along  $\xi_p^\mu = (t, r, 0, 0)$ , which is the projection of  $\xi^A = (t, r, 0, 0, y)$  on  $\Sigma_y$ , i.e.,

$$\mathcal{L}_{\xi_p} h_{\mu\nu} \neq 2h_{\mu\nu}, \quad \text{for } \xi_p^\lambda = (t, r, 0, 0). \quad (43)$$

In fact, we show in the Appendix that there is only one family of homothetic solutions on  $\Sigma_y$ . But the homothetic vector in  $4D$ , say  $\zeta^\mu$ , is *not* parallel to the projected  $\xi_p^\mu$ .

Secondly, although (39) is Riemann-flat, the hypersurfaces  $\Sigma_y$  are curved. Indeed, the non-vanishing components of the Riemann tensor calculated on  $\Sigma_y$  are

$$\begin{aligned} R_{0101} &= \frac{R_{0202}}{r^2} = \frac{R_{0303}}{r^2 \sin^2 \theta} = \frac{(2\lambda_{\xi\xi} + \lambda_\xi^2)e^\lambda}{8y^2}, \\ R_{1212} &= \frac{R_{1313}}{\sin^2 \theta} = \frac{R_{2323}}{r^2 \sin^2 \theta} = -\frac{C e^{3\lambda/2} \lambda_\xi r^2}{4y^2}. \end{aligned} \quad (44)$$

Now, the requirement  $h_{00} \neq 0$  implies  $\lambda_\xi \neq 0$ . Therefore,  $R_{\alpha\beta\mu\nu} \neq 0$ . What this means is that an observer, who is confined to making physical measurements in our ordinary spacetime, can explain the curvature of  $\Sigma_y$  as being produced by (“effective”) matter whose energy-momentum-tensor  $T_{\mu\nu}^{(eff)}$  is given by the Einstein equations in  $4D$ , viz.,

$$G_{\mu\nu} = 8\pi T_{\mu\nu}^{(eff)} \equiv \epsilon \left[ K'_{\mu\nu} + K \left( K_{\mu\nu} - \frac{K}{2} g_{\mu\nu} \right) - 2 \left( K_{\mu\rho} K_\nu^\rho - \frac{1}{4} g_{\mu\nu} K_{\alpha\beta} K^{\alpha\beta} \right) \right], \quad (45)$$

where  $K_{\mu\nu} = (\partial g_{\mu\nu} / \partial y) / 2$ . For the case under consideration, a simple calculation yields<sup>4</sup>

$$8\pi\rho^{(eff)} = \frac{3C}{4y^2} \lambda_\xi e^{-\lambda(\xi)/2}, \quad (46)$$

$$8\pi p^{(eff)} = -\frac{C}{2y^2} \left( \lambda_\xi + \frac{\lambda_{\xi\xi}}{\lambda_\xi} \right) e^{-\lambda(\xi)/2}. \quad (47)$$

We now proceed to illustrate the above discussion with some examples.

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<sup>4</sup>Here  $\rho^{(eff)} \equiv h^{00} T_{00}^{(eff)}$ , and  $p^{(eff)} \equiv -h^{11} T_{11}^{(eff)} = -h^{22} T_{22}^{(eff)} = -h^{33} T_{33}^{(eff)}$ .

#### 4.1 Solution with $\Phi = 1$ : a Riemann-flat embedding for the de Sitter universe

Many authors use the five available degrees of coordinate freedom to set  $g_{4\mu} = 0$  and  $\Phi = 1$ . This is the so-called “Gaussian normal coordinates system” based on  $\Sigma_y$ , where  $n^A$  is taken to be geodesic in  $5D$ . In this system we can easily integrate (32) to obtain (with  $k = 2$ )

$$dS^2 = \frac{1}{\xi^2} dt^2 - \left(B + \frac{C}{\xi}\right)^2 [dr^2 + r^2 d\Omega^2] - dy^2 \quad (48)$$

where  $B$  and  $C$  are dimensionless constants. With the transformation of coordinates<sup>5</sup>

$$\frac{dt}{t} = -\frac{d\tilde{t}}{L}, \quad t \rightarrow e^{-\tilde{t}/L}, \quad (49)$$

where  $L$  is some constant length, (48) becomes

$$dS^2 = \frac{y^2}{L^2} d\tilde{t}^2 - \left(B + \tilde{C}y e^{\tilde{t}/L}\right)^2 [dr^2 + r^2 d\Omega^2] - dy^2, \quad (50)$$

where the new constant  $\tilde{C}$  has dimensions of  $(length)^{-1}$ . On  $\Sigma_y$  the effective matter is given by

$$8\pi\rho^{(eff)} = \frac{3\tilde{C}^2 e^{2\tilde{t}/L}}{(B + \tilde{C}y e^{\tilde{t}/L})^2}, \quad 8\pi p^{(eff)} = -\frac{\tilde{C} e^{\tilde{t}/L} (2B + 3\tilde{C}y e^{\tilde{t}/L})}{y(B + \tilde{C}y e^{\tilde{t}/L})^2}. \quad (51)$$

We note that

$$8\pi\rho^{(eff)} \rightarrow -8\pi p^{(eff)} = \frac{3}{y^2} \quad \text{as } \tilde{t} \rightarrow \infty. \quad (52)$$

Therefore (48), or (50), for  $t \approx 0$  is a Riemann-flat embedding for the de Sitter universe with cosmological “constant”  $\Lambda = 3/y^2$ .

#### 4.2 Solution with $e^\nu = 1$ : a Riemann-flat embedding for Milne’s universe

The choice  $h_{00} = 1$  (and  $h_{0j} = 0$ ) is usual in cosmology; it corresponds to the so-called synchronous reference system where the time coordinate  $t$  is the *proper* time at each point. In such a system the line element (39) becomes

$$dS^2 = dt^2 - (B + Ct)^2 [dr^2 + r^2 d\Omega^2] - \xi^2 dy^2. \quad (53)$$

On  $\Sigma_y$  the effective matter quantities are given by

$$\rho^{(eff)} = -3p^{(eff)} = \frac{3C^2}{8\pi(Bt + Ct)^2}. \quad (54)$$

We note that the equation of state  $\rho = -3p$  appears in different contexts: in Milne’s universe; in discussions of premature recollapse problem [52]; in coasting cosmologies [53]; in cosmic strings [54], [55]; limiting configurations [56]; the exterior spacetime for stellar models in  $5D$  Kaluza-Klein gravity [57].

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<sup>5</sup>Setting  $B = 0$ , and making a coordinate transformation  $dt/t = -\sqrt{\Lambda(\tilde{t})/3} d\tilde{t}$ , where the function  $\Lambda(\tilde{t})$  has units of  $(length)^{-2}$ , (48) can be written as  $dS^2 = y^2 \frac{\Lambda(\tilde{t})}{3} d\tilde{t}^2 - C^2 y^2 e^2 \int \sqrt{\Lambda(\tilde{t})/3} d\tilde{t} [dr^2 + r^2 d\Omega^2] - dy^2$ , which is identical to the  $5D$  line element discussed by Bellini [51].



### 4.3 Riemann-flat embedding for a 4D universe filled with ordinary matter and a cosmological constant

In order to obtain an equation for the unknown function  $\lambda(\xi)$ , let us assume that the effective matter can be separated as

$$\begin{aligned} 8\pi\rho^{eff} &= 8\pi\rho + \Lambda, \\ 8\pi p^{eff} &= 8\pi p - \Lambda, \end{aligned} \quad (55)$$

where  $\rho$  and  $p$  are the density and pressure of the cosmological fluid and  $\Lambda$  is the cosmological constant. In addition, as it is usual in cosmology, we assume that the density and pressure satisfy the barotropic equation of state

$$p = n\rho, \quad (56)$$

where  $n$  is some constant commonly restricted by  $|n| \leq 1$ , which follows from the dominant energy condition [2], [3]. Thus,

$$\rho = \frac{\rho^{eff} + p^{eff}}{(n+1)}, \quad \frac{\Lambda(n+1)}{8\pi} = (n\rho^{eff} - p^{eff}), \quad p = n\rho \quad (57)$$

Using (46) and (47) we find

$$2\lambda_{\xi\xi} + (3n+2)\lambda_{\xi}^2 - h_0\lambda_{\xi}e^{\lambda/2} = 0, \quad \text{with } h_0 \equiv \frac{4\Lambda(n+1)y^2}{C}. \quad (58)$$

where  $h_0$  is a constant. Thus, in general

$$\Lambda = \frac{h_0 C}{4(n+1)y^2}. \quad (59)$$

Setting  $S(\xi) = e^{\lambda(\xi)/2}$ , (58) becomes

$$2SS_{\xi\xi} + 2(3n+1)S_{\xi}^2 - h_0S^2S_{\xi} = 0, \quad (60)$$

whose first integral is

$$S_{\xi} = aS^2 + \frac{b}{S^{(3n+1)}}, \quad \text{where } a \equiv \frac{2\Lambda y^2}{3C}, \quad (61)$$

i.e.,  $h_0 = 6a(n+1)$  and  $b$  is a constant of integration. Thus, in the case under consideration the spacetime part of the 5D metric (39) is given by

$$ds^2 = \frac{2}{C} \left( aS^2 + \frac{b}{S^{(1+3n)}} \right) dt^2 - S^2 [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)]. \quad (62)$$

The corresponding matter quantities are

$$8\pi\rho = \frac{3Cb}{2y^2 S^{3(n+1)}}, \quad p = n\rho, \quad \Lambda = \frac{3Ca}{2y^2}. \quad (63)$$

Let us consider the deceleration parameter  $q$ , which is defined as

$$q = -\frac{S_{\tau\tau}S}{S_{\tau}^2}, \quad (64)$$

where  $S_{\tau} = (1/y)S_{\xi}e^{-\nu/2}$  represents the derivative of the scale factor with respect to the universal time  $\tau$ , which is related to the coordinate time  $t$  by the expression  $d\tau = e^{\nu/2}dt$ . A simple calculation gives,

$$q = \frac{3n+1}{2} - \frac{3a(n+1)}{2[a + b/S^{3(n+1)}]}. \quad (65)$$

In order to simplify this expression let us calculate the density parameters  $\Omega_m = 8\pi\rho/3H^2$  and  $\Omega_\Lambda = \Lambda/3H^2$ , where  $H \equiv S_\tau/S$  is the Hubble parameter. We obtain

$$H^2 = \left(\frac{Ca}{2y^2}\right) \left[1 + \frac{(b/a)}{S^{3(n+1)}}\right], \quad \Omega_m = \frac{(b/a)}{S^{3(n+1)} + (b/a)}, \quad \Omega_\Lambda = \frac{S^{3(n+1)}}{S^{3(n+1)} + (b/a)}. \quad (66)$$

Consequently,

$$S^{3(n+1)} = \left(\frac{\Omega_\Lambda}{\Omega_m}\right) \left(\frac{b}{a}\right). \quad (67)$$

Substituting this expression into (65), and using that  $\Omega_m + \Omega_\Lambda = 1$ , we get

$$q = \frac{3}{2}\Omega_m(n+1) - 1. \quad (68)$$

We note that  $q$  changes sign at  $\Omega_m = \Omega_m^{(crit)} = 2\Omega_\Lambda/(3n+1)$ : for  $\Omega_m < \Omega_m^{(crit)}$  the expansion is slowing down ( $q > 0$ ), for  $\Omega_m > \Omega_m^{(crit)}$  the expansion is speeding up ( $q < 0$ ). Setting  $n = 0$ , in concordance with the fact that our present universe is matter-dominated ( $p = 0$ ), and  $\Omega_m = \Omega_{m|today} \approx 0.3$  we obtain the approximate value of deceleration parameter today, viz.,

$$q_{|today} \approx -0.55, \quad (69)$$

which is consistent with observations [4], [6].

#### 4.3.1 Analysis of the evolution equation (61)

Let us notice that, in general, (61) cannot be integrated in terms of elementary functions. However the approximate behavior of the solution is as follows. In the very early universe, when  $S \approx 0$ , the second term in the r.h.s. of (61) dominates over the first one. Thus, we find  $S \sim \xi^{1/(3n+2)}$  which implies that the actual behavior of the scale factor depends on the equation of state.

**False vacuum:  $n = -1$ .** Following Zeldovich [7], one may imagine that after the spontaneous birth (where the spacetime separates from the extra dimension) our universe enters a de Sitter phase of expansion. In this phase  $n = -1$  and  $S \sim 1/t$  on every hypersurface  $\Sigma_y$ , which means that  $S = 0$  at coordinate time  $t = \infty$ . With the transformation of coordinates  $t \rightarrow e^{-\omega\tau}$ , where  $\omega$  is a constant, we obtain  $S \sim e^{\omega\tau}$ . Thus,  $S = 0$  at  $\tau = -\infty$  which corresponds to  $t = \infty$ . Thus, during the inflationary period the expansion of the universe is described by

$$ds^2 = d\tau^2 - Ae^{2\sqrt{\Lambda_{(infl)}/3}\tau} [dr^2 + r^2 d\Omega^2], \quad (70)$$

where  $\Lambda_{(infl)} = 3\omega^2$  is the “effective” cosmological constant in this epoch. According to (63) it is related to the present cosmological constant by

$$8\pi\rho^{(eff)} = \Lambda_{(infl)} = \Lambda \left(1 + \frac{b}{a}\right). \quad (71)$$

However, it should be noted that the de Sitter solution is unstable under small perturbations. Therefore, it is impossible to extrapolate it to  $\tau = -\infty$  [7]. What this means is that (70) does not describe  $S = 0$ , which is singular on  $\Sigma_y$  (but non-singular in  $5D$ ), corresponding to the moment of spontaneous birth of our universe<sup>6</sup>.

**FRW evolution.** The switch from the de Sitter exponential expansion to the radiation dominated FRW universe is a jump of pressure from  $p = -\rho$  to  $p = \rho/3$ . In this case,  $S \sim t^{1/3}$  on every  $\Sigma_y$ . Now changing  $t \rightarrow \tau^{3/2}$  the scale factor becomes  $S \sim \tau^{1/2}$ , which is the usual expression for a radiation dominated era. For any  $n \neq -1$  the transformation  $t \rightarrow \tau^{2(3n+2)/[3(1+n)]}$  allows us to recover the familiar spatially-flat FRW models, viz.,

$$ds^2 = d\tau^2 - B\tau^{4/[3(1+n)]} [dr^2 + r^2 d\Omega^2]. \quad (72)$$

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<sup>6</sup>Models for inflation estimate that the de Sitter phase of exponential expansions begins at  $\tau = 10^{-42}$  s [58].

Let us call  $\tau_0$  the moment of transition from the de Sitter phase to the relativistic plasma ( $n = 1/3$ ) FRW solution. A jump in the pressure indicates that the second derivative of the scale factor is discontinuous, which in turn implies a jump in the deceleration parameter  $q$ ; in the present case from  $q = -1$  to  $q = 1$ . However, in order to guarantee the continuity of energy density and the Hubble term, the scale factor and its first derivative must be continuous during the jump<sup>7</sup>. This allows us to relate  $\tau_0$  and  $\Lambda_{(infl)}$ , namely,

$$\tau_0 = \frac{1}{2} \sqrt{\frac{3}{\Lambda_{(infl)}}}. \quad (73)$$

Our toy model does not provide any mechanism for calculating the time of transition, neither is the object of the present work, however various inflationary models suggest that inflation ends at  $\tau_0 = 10^{-32 \pm 6}$  s [58]. Using this number as a reference we obtain

$$\Lambda_{(infl)} \sim 10^{48 \pm 12} \text{ cm}^{-2}, \quad (74)$$

which is huge compared to the present value of the cosmological constant  $\Lambda \sim 10^{-51} \text{ cm}^{-2}$ , measured in the current  $\Lambda$ CDM model of cosmology [4]-[6]. Thus,

$$\Lambda_{(infl)} \approx 10^{99 \pm 12} \Lambda. \quad (75)$$

**Accelerated expansion.** As the universe expands the deceleration parameter  $q$ , which is given by (65), changes sign again at

$$S_{(q=0)} = \left[ \frac{3n+1}{2} \left( \frac{\Lambda_{(infl)}}{\Lambda} - 1 \right) \right]^{1/3(n+1)}, \quad (76)$$

where we have used (71) to express the ratio  $(b/a)$ . This expression uncovers two interesting physical concepts: (i) if  $\Lambda_{(infl)} = \Lambda$ , then the universe would have never changed from decelerated to accelerated expansion, and (ii) the very large ratio  $\Lambda_{(infl)}/\Lambda$ , evaluated in (75), explains the very large radius of the universe.

The present value of  $q$ , as calculated in (69), is approximately  $-0.55$  which means that our universe is in a phase of accelerated expansion. At late times, for large values of  $S$ , ( $S > S_{(q=0)}$ ), the first term in (61) starts dominating over the second one. Asymptotically, for  $S \gg S_{(q=0)}$ , in terms of universal time  $\tau$  our world enters a new era of exponential expansion with  $q = -1$  and

$$ds^2 = d\tau^2 - e^{2\sqrt{\Lambda/3}\tau} [dr^2 + r^2 d\Omega^2]. \quad (77)$$

In summary, the toy cosmological models considered in this section allow us to conclude that the Riemann-flat embeddings given by (32) are rich enough as to accommodate the essential features of the evolution of our universe.

## 5 Concluding remarks

In this paper we have found the most general self-similar, Ricci-flat, homogeneous and isotropic cosmologies in  $5D$ . Self-similarity requires that all the dimensionless quantities in the theory be functions of a single variable  $\xi$ , which in the case studied here is some combination of  $t$  and  $y$  (9). As a consequence of the symmetry, and the field equations  $R_{AB} = 0$ , we have found that (i) the three-dimensional spatial surfaces defined by  $t = \text{constant}$  and  $y = \text{constant}$  have to be flat (14); (ii) the self-similar variable can be taken either as  $\xi = t/y$  (32), which is the more general case, or as  $\xi = e^{\alpha t}/e^{\alpha y}$  (34); (iii) the extra dimension must be spacelike (30). Then, we obtained the general solutions of the field equations in terms of one arbitrary function of  $\xi$  (32)-(34).

These solutions are Riemann-flat in  $5D$  but curved in  $4D$  (44). This latter property, together with the property (i) mentioned above, allow us to interpret these cosmologies as Riemann-flat embeddings for spatially-flat FRW cosmologies in  $4D$ . In this interpretation our universe arises as a spontaneous separation from an empty  $5D$  Minkowski space, as envisioned by Zeldovich [7].

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<sup>7</sup>This is equivalent to requiring continuity of the first and second fundamental forms across a  $3D$  surface  $\tau = \tau_0 = \text{constant}$  [45].

We have analyzed in some detail the  $4D$  interpretation of (32) (for  $l = 2$ ), which admits the simply homothetic Killing vector  $\xi^A = (t, r, 0, 0, y)$ . We have seen that the  $4D$  projection of this vector does not constitute a  $4D$  homothetic Killing vector for the metric induced on  $\Sigma_y$  (43). For the physics in  $4D$  this opens a wide range of non-homothetic, but still self-similar, possibilities for the evolution of our universe (A-1)-(A-3), among them it allows the introduction of a cosmological constant and a de Sitter phase of exponential expansion.

The question is how to establish the arbitrary function of  $\xi$  in (32)-(34). There are many plausible ways for selecting it. As an example one can choose some “geometrical” criteria, e.g., choose a particular coordinate frame or to assume some specific symmetry, as we do in sections 4.1, 4.2 and the Appendix, respectively. A distinct approach, which is probably more satisfactory from a “physical” point of view, is to select the arbitrary function by imposing conditions on the effective picture in  $4D$ ; this is what we do in section 4.3. Certainly, the existence of an arbitrary function allows us to accommodate a number of physical models.

The model discussed in 4.3 roughly presents the essential features of the evolution of our universe. Specifically, it shows inflation from an original birth, not described by the de Sitter solution (70), followed first by a FRW phase and then by an accelerated expansion. The effective cosmological constants during inflation  $\Lambda_{(infl)}$  and after inflation  $\Lambda$  are not equal, otherwise the universe cannot enter a phase of accelerated expansion (76).

In summary, in this work we have studied a family of self-similar cosmological metrics in  $5D$ . As far as the author knows, this is the first work where self-similarity is used for finding exact solutions to the field equations in  $5D$ . The solutions may be used as  $5D$  embeddings for the spatially-flat FRW cosmological models of ordinary general relativity in  $4D$ .

We have not investigated fully the possible physical interpretations of (32), we have just considered a particular case. Neither have we investigated the solution (34), which seems to be totally different from (32). An immediate extension of this work is the study of self-similar cosmologies with extra dimensions *without* the assumption of spatial isotropy and/or homogeneity. Also, it is interesting to find the transformation of coordinates from (32)-(34) to  $\mathcal{M}_5$  given by (36). This is a non-trivial task, but the existence of such a transformation is guaranteed by the fact that the Riemann tensor in  $5D$  vanishes. With the appropriate transformation of coordinates at hand one can extend and generalize previous investigations [28] and study the birth of the universe in more detail.

## Appendix: Homothetic symmetry on $\Sigma_y$

Our aim here is (i) to show that the requirement of homothetic symmetry on  $\Sigma_y$  singles out one specific metric in  $4D$ , namely, the spacetime part of the  $5D$  metric (5), and that (ii) the homothetic vector is not parallel to  $\xi_p^\mu$  (43).

In order to do this, let us express the induced metric on  $\Sigma_y$ , which is given by (42), in terms of  $S = e^{\lambda/2}$ ,

$$ds^2 = h_{\mu\nu} dx^\mu dx^\nu = \frac{2S_\xi}{C} dt^2 - S^2(\xi) [dr^2 + r^2 d\Omega^2]. \quad (\text{A-1})$$

The Lie derivative of this metric along the  $4D$  vector

$$\zeta^\mu = (At, Br, 0, 0), \quad (\text{A-2})$$

where  $A$  and  $B$  are some constants, is given by

$$\mathcal{L}_\zeta h_{00} = 2h_{00} A \left( 1 + \frac{\xi S_{\xi\xi}}{2S_\xi} \right), \quad \mathcal{L}_\zeta h_{ij} = 2h_{ij} \left[ A \frac{\xi S_\xi}{S} + B \right] \quad (\text{A-3})$$

Thus, the requirement

$$\mathcal{L}_\zeta h_{\alpha\beta} = 2h_{\alpha\beta}, \quad (\text{A-4})$$

generates two independent equations, viz.,

$$\frac{S_{\xi\xi}}{S_\xi} = \frac{2(1-A)}{A\xi}, \quad \text{and} \quad \frac{S_\xi}{S} = \frac{(1-B)}{A\xi}. \quad (\text{A-5})$$

From the first equation we get  $S = C_1 \xi^{(2-A)/A} + C_2$ , while from the second one we obtain  $S = C_3 \xi^{(1-B)/A}$ , where  $C_1$ ,  $C_2$  and  $C_3$  are constants of integration. Compatibility of these expressions demand  $(A - B) = 1$ , and  $C_2 = 0$ . Consequently,  $S \sim \xi^{(2-A)/A}$  and  $e^\nu \sim \xi^{2(1-A)/A}$ .

Since  $A$  is an arbitrary parameter, without loss of generality we can set

$$A = \frac{2\alpha - 1}{\alpha}. \quad (\text{A-6})$$

With this selection the metric in  $4D$  becomes

$$ds^2 = A \xi^{2(1-\alpha)/(2\alpha-1)} dt^2 - B \xi^{2/(2\alpha-1)} [dr^2 + r^2 d\Omega^2], \quad (\text{A-7})$$

which is identical to the spacetime part of (5) and admits the homothetic Killing vector

$$\zeta^\mu = \left( \frac{2\alpha - 1}{\alpha} t, \frac{\alpha - 1}{\alpha} r, 0, 0 \right). \quad (\text{A-8})$$

Clearly, this vector is not parallel to  $\xi_p^\mu = (t, r, 0, 0)$ . We emphasize that the  $5D$  metric (4) is homothetic along the  $5D$  vector  $\xi^A = (t, r, 0, 0, y)$ , but its spacetime part is homothetic along (A-8) and not along  $\xi_p^\mu = (t, r, 0, 0)$ .

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